

# Spherical, Oscillatory $\alpha^2$ -Dynamo Induced by Magnetic Coupling Between a Fluid Shell and an Inner Electrically Conducting Core: Relevance to the Solar Dynamo

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## ABSTRACT

A two-layer spherical  $\alpha^2$ -dynamo model consisting of an inner electrically conducting core (magnetic diffusivity  $\lambda_i$  and radius  $r_i$ ) with  $\alpha = 0$  surrounded by an electrically conducting spherical shell (magnetic diffusivity  $\lambda_o$  and radius  $r_o$ ) with a constant  $\alpha$  is shown to exhibit oscillatory behavior for values of  $\beta = \lambda_i/\lambda_o$  and  $r_i/r_o$  relevant to the solar dynamo. Time-dependent dynamo solutions require  $r_i/r_o \geq 0.55$  and  $\beta \leq O(1)$ . For the Sun,  $r_i/r_o$  is about 0.8 and  $\beta \approx 10^{-3}$ . The time scale of the oscillations matches the 22 year period of the sunspot cycle for  $\lambda_0 = O(10^2 \text{ km}^2 \text{ s}^{-1})$ . It is unnecessary to hypothesize an  $\alpha\omega$ -dynamo to obtain oscillatory dynamo solutions; an  $\alpha^2$ -dynamo suffices provided the spherical shell region of dynamo action lies above a large, less magnetically diffusive core, as is the case for the solar dynamo.

*Subject headings:* convection — hydrodynamics — instabilities — magnetic fields — Sun: magnetic fields — stars: magnetic fields

## 1. Introduction

$\alpha^2$ - and  $\alpha\omega$ -dynamo models have been studied for decades (Braginsky 1964; Steenbeck and Krause 1966; Roberts 1972; Moffatt 1978; Gubbins and Roberts 1987; Baryshnikova and Shukurov

1987). The  $\alpha^2$ -dynamo is usually stationary (e.g., Roberts 1972; Gubbins and Roberts 1987; Hollerbach 1996), although oscillatory  $\alpha^2$ -dynamos have been found to occur in the special circumstance wherein  $\alpha$  changes rapidly in boundary layers (Radler and Brauer 1987; Baryshnikova and Shukurov 1987). In such cases, the period of the  $\alpha^2$ -dynamo depends strongly on the location of the  $\alpha$ -boundary layer and is typically an order of magnitude or more smaller than the magnetic diffusion time across the dynamo generation region. In general, oscillatory dynamo behavior has been produced by combination of the  $\alpha$ - and  $\omega$ -effects. Kinematic models of the solar dynamo, which is inherently oscillatory, have been of the  $\alpha\omega$ -type (e.g., Roald and Thomas 1997).

In this paper, we report that spherical oscillatory  $\alpha^2$ -dynamos can be simply induced by the magnetic coupling between an electrically conducting outer fluid shell and a conducting inner spherical core even when  $\alpha$  in the outer shell is a constant. The period of oscillation is of the same order of magnitude as the magnetic diffusion time across the outer shell and depends largely on the electrical conductivity of the inner core. Oscillatory behavior occurs when the outer region of dynamo action surrounds a large, less magnetically diffusive core. The radiative interior of the Sun is a large region with a smaller magnetic diffusivity than the overlying convection zone wherein dynamo action occurs. The oscillatory character of the Sun’s magnetic field, as expressed in the 22 year periodicity of the sunspot cycle, could then be related to the electromagnetic coupling of the region of magnetic field generation in the convection zone with the radiative core of the Sun. The importance of an inner electrically conducting core to the problem of magnetic field generation in an overlying spherical shell has been emphasized in the  $\alpha^2$ -type models of the geodynamo by Hollerbach and Jones (1993, 1995), Hollerbach (1996), and Gubbins (1999). Nevertheless, the effects of an inner core, with magnetic diffusivity different from that of the overlying fluid convecting shell in which dynamo action takes place, have not been fully elucidated.

The problem of inner core-fluid shell coupling is made difficult by complicated electromagnetic matching conditions at the interface between the regions. For this reason, and to facilitate understanding of the physical effects, we consider the simplest type of  $\alpha^2$ -dynamo model consisting of a spherical shell with  $\alpha = \text{constant}$  surrounding a core with  $\alpha = 0$ . We derive the appropriate matching conditions for a core and shell of arbitrary magnetic diffusivity. These matching conditions do not appear to have been considered in previous studies of spherical  $\alpha^2$ -dynamos and they result in oscillatory dynamo solutions.

## 2. Model, Equations and Boundary Conditions

The model consists of a turbulent fluid spherical shell of inner radius  $r_i$  and outer radius  $r_o$  with constant (turbulent) magnetic diffusivity  $\lambda_o$ . A magnetic field is generated in the shell by the  $\alpha$ -effect (Steenbeck and Krause 1966; Roberts 1972). For  $r > r_o$ , we assume there is a non-conductor; for  $r < r_i$  we assume that there is a conductor with magnetic diffusivity  $\lambda_i$ . The kinematics of the  $\alpha^2$ -dynamo in the spherical shell is governed by the non-dimensional linear equations for the

magnetic field  $\mathbf{B}_o$

$$\frac{\partial \mathbf{B}_o}{\partial t} = R(1 - \eta) \nabla \times \alpha \mathbf{B}_o + \nabla^2 \mathbf{B}_o \quad (1)$$

$$\nabla \cdot \mathbf{B}_o = 0 \quad (2)$$

In the inner sphere the magnetic field  $\mathbf{B}_i$  is governed by

$$\frac{\partial \mathbf{B}_i}{\partial t} = \beta \nabla^2 \mathbf{B}_i \quad (3)$$

$$\nabla \cdot \mathbf{B}_i = 0 \quad (4)$$

Equations (1)–(4) are scaled by the thickness of the shell  $(r_o - r_i)$  and by the magnetic diffusion timescale  $(r_o - r_i)^2 / \lambda_o$ . The scaling of the linear system of equations for the magnetic field is arbitrary. The non-dimensional parameters in the above equations,  $\beta$ ,  $\eta$ , and the magnetic Reynolds number  $R$  are defined as

$$\beta = \frac{\lambda_i}{\lambda_o}, \quad \eta = \frac{r_i}{r_o}, \quad R = \frac{r_o \alpha}{\lambda_o} \quad (5)$$

Since the main purpose of this paper is to understand the effect of an electrically conducting inner core, we adopt the simplest possible model and take  $\alpha$  constant in the spherical shell  $r_i < r < r_o$ ;  $\alpha$  is zero outside the shell. With this assumption, spherical harmonics are decoupled and the problem is reduced to a one-dimensional problem with complicated boundary conditions.

At the interface between the shell and the perfectly insulating exterior, i.e., at  $r = r_o$ , the magnetic field must be continuous

$$\mathbf{B}_o = \mathbf{B}^{(\epsilon)} \quad \text{at} \quad r = r_o \quad (6)$$

where  $\mathbf{B}^{(\epsilon)} = -\nabla\phi$  is the magnetic field in the insulating exterior  $r > r_o$ , and  $\nabla^2\phi = 0$ . At the interface between the shell and the conducting inner sphere, i.e., at  $r = r_i$ , both the magnetic field  $\mathbf{B}$  and tangential components of the electric field  $\mathbf{E}$  must be continuous

$$\mathbf{B}_o = \mathbf{B}_i, \quad \hat{\mathbf{r}} \times \mathbf{E}_o = \hat{\mathbf{r}} \times \mathbf{E}_i \quad \text{at} \quad r = r_i \quad (7)$$

where  $\hat{\mathbf{r}}$  is the unit radial vector,  $\mathbf{E}_o$  is the electric field in the outer shell, and  $\mathbf{E}_i$  is the electric field in the inner core.

Conditions (2) and (4) allow us to express the magnetic fields as a sum of poloidal and toroidal vectors

$$\mathbf{B}_o = \nabla \times \nabla \times \mathbf{r} h_o + \nabla \times \mathbf{r} g_o \quad (8)$$

$$\mathbf{B}_i = \nabla \times \nabla \times \mathbf{r} h_i + \nabla \times \mathbf{r} g_i \quad (9)$$

where  $\mathbf{r}$  is the position vector. Use of Equation (8) in boundary condition (6) and expansion of  $h_o$  and  $g_o$  in terms of spherical harmonics give

$$g_o = 0, \quad \frac{\partial h_o}{\partial r} + \frac{(l+1)h_o}{r} = 0 \quad \text{at} \quad r = r_o \quad (10)$$

where  $l$  is the degree of the spherical harmonic  $Y_l^m$ .

Extra care must be taken for the magnetic boundary conditions at the interface  $r = r_i$ . There are four different cases that we have studied:

(I). The limit  $\beta \rightarrow \infty$  for both stationary and oscillatory dynamos. In this case, the boundary condition for the magnetic field is simply

$$g_o = 0, \quad \frac{\partial h_o}{\partial r} - \frac{lh_o}{r} = 0 \quad \text{at } r = r_i \quad (11)$$

(II). The limit  $\beta \rightarrow 0$  for a stationary dynamo. In this case, boundary conditions (7) require

$$h_o = 0, \quad R(1 - \eta) r \frac{\partial h_o}{\partial r} - \frac{\partial(r g_o)}{\partial r} = 0 \quad \text{at } r = r_i \quad (12)$$

(III). The limit  $\beta \rightarrow 0$  for an oscillatory dynamo. In this case, boundary conditions (7) require

$$\frac{\partial h_o}{\partial r} - \frac{lh_o}{r} = 0, \quad R(1 - \eta)(l + 1)h_o - \frac{\partial(r g_o)}{\partial r} = 0 \quad \text{at } r = r_i \quad (13)$$

(IV). The general case for  $\beta$  not tending toward 0 or  $\infty$ , for both stationary and oscillatory dynamos. In this case, boundary conditions (7) require

$$g_o = g_i, \quad h_o = h_i, \quad \frac{\partial h_o}{\partial r} = \frac{\partial h_i}{\partial r}, \quad R(1 - \eta) \frac{\partial(r h_o)}{\partial r} - \frac{\partial(r g_o)}{\partial r} + \beta \frac{\partial(r g_i)}{\partial r} = 0 \quad \text{at } r = r_i \quad (14)$$

The last case is evidently the most complicated one. The solutions presented below show that for  $\beta \geq 10$ , case I provides a good approximation to case IV, while for  $\beta \leq 0.1$ , cases II and III provide a good approximation to case IV. In cases II through IV,  $g$  and  $h$  are coupled by boundary conditions (7). The solutions are invariant to a change in the sign of  $R$ .

### 3. Solution Method

In all cases, solutions are expanded in terms of spherical harmonics, implicit in the forms of the boundary and interface conditions given above. The spherical harmonics are decoupled and only the lowest one ( $\ell = 1$ ) is used in the analysis. The  $\ell = 2$  mode, not discussed here, behaves similarly to the  $\ell = 1$  mode.

The time dependence of the solutions is written as  $\exp(\sigma_r + i\omega t)$  and onset of dynamo action ( $\sigma_r = 0$ ) is sought. As discussed below, dynamos are either stationary ( $\omega = 0$ ) or oscillatory ( $\omega \neq 0$ ). The frequency  $\omega$  is dimensionless with respect to the timescale  $(r_o - r_i)^2 / \lambda_o$ .

In case I, a perfectly insulating core, and in cases II and III, a perfectly conducting core, it is only necessary to solve for  $g_o$  and  $h_o$  subject to the above boundary conditions at  $r_i$  and  $r_o$ . Exact analytic solutions for  $g_o$  and  $h_o$  in these cases can be found in terms of the spherical Bessel functions of the first and second kind. The solutions reduce to finding the eigenvalues of a  $4 \times 4$

matrix. The eigenvalues give critical values of the magnetic Reynolds number  $R$  as a function of  $\eta = r_i/r_o$ , for which steady or oscillatory dynamos are possible (i.e.,  $\sigma_r = 0$ ).

In case IV, a core of arbitrary  $\beta$ , solutions must be obtained in both the shell and the core subject to the above matching conditions, i.e.,  $g_o$ ,  $h_o$ ,  $g_i$ , and  $h_i$  must be determined. In principle, analytic solutions are possible, but it is computationally more efficient to seek numerical solutions. We do this by employing a spectral-Tau method which expands solutions in terms of Chebyshev polynomials. The numerical solutions for arbitrary  $\beta$ , and the analytic solutions determined independently in cases I, II, and III, provide a mutual validation of the separate methods. For appropriate values of  $\beta$ , the solutions of the separate methods agree essentially exactly.

#### 4. Results

The principal results of this study are summarized in Figure 1 which gives the critical magnetic Reynolds number  $R_{cr}$  for the onset of dynamo action in the dipole ( $l = 1$ ) mode as a function of  $\eta$ , the ratio of the inner radius of the shell to its outer radius. The critical value of  $R$  increases with increasing  $\eta$ . For an insulating core ( $\beta \gg 1$ , dashed curve) dynamo solutions are always

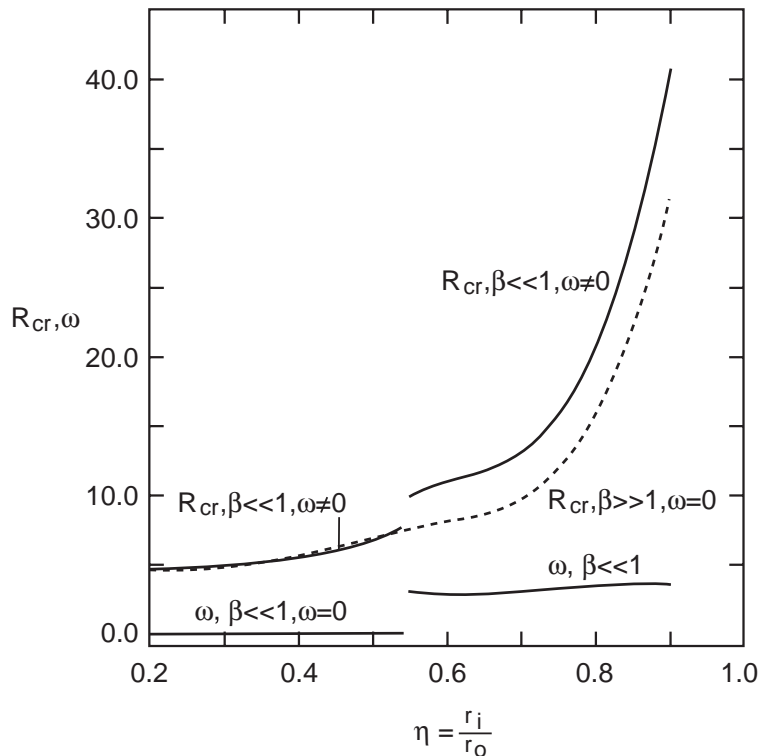


Fig. 1.— Critical magnetic Reynolds number  $R_{cr}$  and dimensionless frequency  $\omega$  of dipolar dynamo solutions at the onset of dynamo action vs.  $\eta$ .

steady ( $\omega \equiv 0$ ). For a perfectly conducting core ( $\beta \ll 1$ , solid curve) there are two branches of dynamo solutions depending on  $\eta$ ; for  $\eta \leq 0.55$  the dynamo is steady, but for  $\eta \geq 0.55$  the dynamo is oscillatory. There is a jump in  $R_{cr}$  at the transition from steady dynamos to oscillatory dynamos near  $\eta = 0.55$ . The dimensionless frequency  $\omega$  of the oscillatory dynamos, also shown in Figure 1 as a function of  $\eta$ , varies between about 2.5 and 3 for all values of  $\eta$  considered.

The values of  $R_{cr}$  for arbitrary  $\beta$  lie in the narrow space between the solid and dashed curves of Figure 1. Importantly, it is found that  $\beta$  need not in fact be very small compared with unity for oscillatory dynamos to exist. For example, when  $\eta = 0.8$ , the value appropriate to the solar dynamo, oscillatory dynamo solutions are found for  $\beta$  less than 2 to 3.

## 5. Discussion

The problem solved above is a classically simple one of the type considered by Steenbeck and Krause (1966) and Roberts (1972) decades ago. Yet the effects of an inner electrically conducting core on the  $\alpha^2$ -dynamo are subtle and not heretofore appreciated. They enter through the complicated electromagnetic matching conditions at the interface between the core and the surrounding shell in which dynamo action occurs. The main effect of the core is to introduce time dependence into the dynamo solutions for cores whose radii are greater than about 0.55 of the outer radius of the shell. An additional requirement for time dependence is that the core be a reasonably good electrical conductor; in terms of the magnetic diffusivity ratio  $\beta = \lambda_i/\lambda_o$ ,  $\beta \leq O(1)$  suffices for oscillatory dynamo behavior.

The importance of all this to the solar dynamo is that the parameters of the solar dynamo satisfy the requirements of oscillatory  $\alpha^2$ -dynamo solutions. For the solar dynamo  $\eta$  is about 0.8 and  $\beta$  is about  $10^{-3}$  (Moffatt 1978). In addition, if  $\omega$  (from Figure 1) is made dimensional using the time scale  $(r_o - r_i)^2/\lambda_o$  with  $r_o - r_i = 1.4 \times 10^5$  km and  $\lambda_o = O(10^2 \text{ km}^2 \text{ s}^{-1})$  (eddy magnetic diffusivity), then the period of the oscillatory dynamo solution is comparable to the 22 year period of the sunspot cycle. Thus,  $\alpha^2$ -dynamo action alone could be responsible for the observed time dependence of the large scale solar magnetic field. It is not our intent to suggest that the  $\omega$ -effect is not significant in dynamo action in general, or in the solar dynamo in particular, because it represents a physically important process. Our purpose is only to clarify some physics and demonstrate the potential importance of a hitherto overlooked effect, that of the oscillatory  $\alpha^2$ -dynamo. A detailed analysis of the cases  $\alpha = \alpha(r)$  and  $\alpha \propto \cos \theta$  ( $\theta$  = polar angle) is in progress.

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